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An alternative method for analysing buckling of laminated composite beams

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Scientific paper - Preliminary note

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An alternative method for analysing buckling of laminated composite beams

In this study, two new functionals are derived based on Gâteaux differential in order to analyse buckling of symmetric cross-ply laminated composite straight beams. The functional comprises four independent variables, i.e. deflection, rotation, shear force and bending moment for Timoshenko beam, and two independent variables, deflection and bending moment, for Euler-Bernoulli beam. The application possibilities and performance of the proposed mixed finite element formulation are presented on several numerical examples.

Key words:
Gâteaux differential, composite straight beams, mixed finite element formulation, buckling analysis, cross-ply laminated composite

Prethodno priopćenje

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Alternativna metoda za analizu izvijanja lameliranih kompozitnih greda

U radu su primjenom Gâteauxove derivacije izvedena dva nova funkcionala kako bi se provela analiza izvijanja simetričnih križno uslojenih ravnih lameliranih kompozitnih greda. Funkcional sadrži četiri nezavisne varijable, a to su progib, rotacija, poprečna sila i moment savijanja za Timošenkovu gredu, te dvije nezavisne varijable, progib i moment savijanja, za Euler-Bernoullijevu gredu. Mogućnost primjene i učinkovitost predložene mješovite formulacije konačnih elemenata prikazana je na nekoliko numeričkih primjera.

Ključne riječi:
Gâteauxova derivacija, ravne kompozitne grede, mješovita formulacija konačnih elemenata, analiza izvijanja, križno uslojeni lamelirani kompozit

Vorherige Mitteilung

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Alternative Methode zur Auswertung der Durchbiegung von lamellierten Kompositbalken

In der Arbeit wurden anhand der Gâteaux-Ableitung zwei neue Funktionale zur Auswertung der Durchbiegung von geraden lamellierten Kompositbalken aus Kreuzschichtholz abgeleitet. Das Funktional enthält vier unabhängige Variablen: Durchbiegung, Rotation, Querkraft und Biegemoment für den Timoshenko-Balken und zwei unabhängige Variablen, die Durchbiegung und das Drehmoment, für den Euler-Bernoulli-Balken. Die Möglichkeiten der Anwendung und die Effizienz der vorgeschlagenen gemischten Formulierung von finiten Elementen wurden anhand von einigen numerischen Beispielen dargestellt.

Schlüsselwörter:
Gâteaux-Ableitung, gerade Kompositbalken, gemischte Formulierung von finiten Elementen, Verformungsanalyse, Kreuzschichtlamina

1. Introduction

Composite materials are extensively used in different branches of engineering because of their high strength/weight and stiffness/weight ratios. Mechanical behaviour of composite materials has been of interest to many researchers [1-3]. The issue of understanding buckling behaviour of laminated structural components has been gaining considerable attention in recent times. Buckling analysis of symmetric cross-ply laminated composite beams is conducted in this paper. Many calculation models can be found in literature for buckling analysis of laminated composite beams. Analytical or numerical methods have been employed to find appropriate solutions. The most frequently applied numerical methods are Rayleigh-Ritz and finite element methods.

By using different higher-order shear beam theories, the buckling and vibration analysis of cross-ply and angle-ply laminated composite beams is conducted in [4] for various boundary conditions employing the Ritz method. Adopting the dynamic stiffness method, the free vibration and buckling behaviour of axially loaded laminated composite beams with arbitrary lay-up is studied in [5-7]. Based on the two dimensional theory, the free vibration and buckling analysis of composite beams with interlayer slip in line is analysed in [8]. An exact solution for the post-buckling and free vibrations of a symmetrically laminated composite beam with different boundary conditions is presented in [9]. Analytical solutions for the free vibration and buckling of cross-ply composite beams with arbitrary boundary conditions are developed in [10] and [11] in conjunction with the state space approach. Based on a higher order shear deformation theory that assumes nonlinear variation of displacement field, a single layer beam finite element model is proposed in [12] for studying the buckling behaviour of anisotropic sandwich beams. Analytical solutions for static, dynamic and buckling analysis of composite beams are presented in [13] based on the Timoshenko beam theory. A buckling analysis of simply supported composite laminated beams is presented in [14] based on the modified couple stress theory by applying the minimum potential energy principle, and taking into account Euler-Bernoulli and Timoshenko beam theories. By using the method of power series expansion of displacement components, natural frequencies and buckling stresses of simply supported laminated composite beams are evaluated in [15] based on the higher order beam theory. Analytical solutions for the free vibration and buckling behaviour of laminated composite and sandwich beams are developed in [16]. Using the refined shear deformation theory, the vibration and buckling analysis of cross-ply composite beams is presented in [17] by means of a displacement based finite element method. The buckling behaviour of the laminated composite beam and flat panels is analysed in [18] using the 1D finite element formulation within the framework of the Carrera Unified Formulation. Based on the shear deformation theory, the vibration and buckling behaviour of composite beams with arbitrary lay-ups is studied in [19] using a displacement based finite element method. A buckling analysis of two-layer composite beams is performed in [20] by means of the displacement based finite element method using the Reddy's higher order beam theory.

A considerable research has been conducted on the buckling of laminated composite beams. However, to the best of the authors' knowledge, the buckling analysis of laminated composite beams using the mixed finite element (MFE) method has not as yet been reported. The application of an efficient and simple method to the buckling analysis of symmetric cross-ply laminated composite straight beams will be presented in this study.

In this research based on the Euler-Bernoulli beam theory and Timoshenko beam theory, two new functionals are constructed using a systematic procedure based on the Gâteaux differential, as a means to analyse the buckling problem of symmetric cross-ply laminated straight beams. Some advantages of the Gâteaux differential approach, first used in [21] to obtain a functional, can be presented as follows:

- All field equations can systematically be enforced to the functional
- Boundary conditions of the problem can easily be obtained
- Field equations can be checked out with potential test
- Developed mixed element completely eliminates the shear locking phenomenon.

Based on the Gâteaux Differential, free vibration of laminated composite curved beams is studied in [22]. In addition, the free vibration analysis of cross-ply and angle-ply laminated composite beams is conducted in [23, 24]. Furthermore, this powerful method has recently been applied in [25] for the analysis of cross-ply laminated composite thick plates.

New mixed type finite elements are formulated in this study. A Timoshenko beam finite element has two nodes and four degrees of freedom per node, while an Euler-Bernoulli beam finite element has two nodes and two degrees of freedom per node. A finite element analysis program has been developed, and the buckling analysis of symmetric cross-ply laminated composite beams with different boundary conditions, lamination variables such as ply orientations and stacking sequences, geometrical and material properties, is performed using the proposed finite element program. The validity of the presented MFE formulation is proven by comparing the results with literature information. Numerical comparison shows that the results of this research agree well with research results presented in relevant literature.

2. Field Equations and functional

A symmetric laminated composite straight beam in a Cartesian coordinate system, denoted by xyz as shown in Figure 1, is considered for a length L, thickness h, and width b.

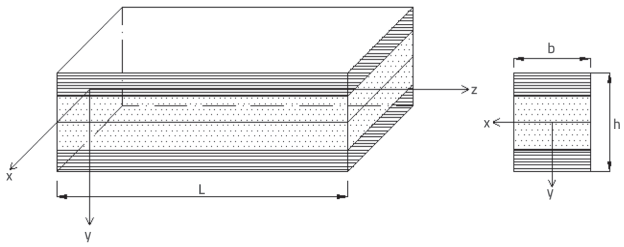

Figure 1. Geometry of symmetric laminated straight beam

Consider a uniformly loaded bar, with positive directions of internal forces as shown in Figure 2, where q_z is the uniformly distributed axial load, N is the axial force, H is the horizontal force, and M is the bending moment.

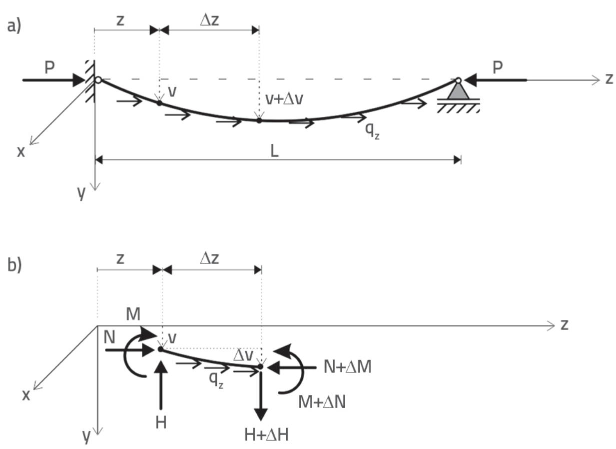

Figure 2. a) Initial straight state and buckled state of a bar; b) Free-body segment of a bar

Based on the Euler-Bernoulli beam theory assumption, the moment equilibrium equation about the upper end of the bar, when q_z is set to zero, can be obtained as follows,

$$\frac{d^2M}{dz^2} = N\frac{d^2v}{dz^2} \tag{1}$$

and the bending moment becomes:

$$\frac{d^2v}{dz^2} = -\frac{M}{D_x} \tag{2}$$

where v is the displacement along the y-axis and D_x is the bending rigidity of the beam, expressed as follows:

$$D_x = \sum_{k=1}^{n} \bar{Q}_{11}^{(k)} I^{(k)} \tag{3}$$

By considering the Timoshenko beam theory, field equations can be given in the following form:

$$-N\frac{d^2v}{dz^2} + \frac{dT}{dz} = 0 \tag{4}$$

$$\frac{dM}{dz} - T = 0 \tag{5}$$

$$-\frac{d\Omega}{dz} + \frac{M}{D_x} = 0 \tag{6}$$

$$-\frac{dv}{dz} - \Omega + \frac{T}{C_y} = 0 \tag{7}$$

where T is the conservative shear force, Ω is the cross-sectional rotation about x-axis and C_y is the shear rigidity of the beam given by:

$$C_y = \sum_{k=1}^{n} \bar{Q}_{66}^{(k)} A^{(k)} \tag{8}$$

In Eqs. (3) and (8), k represents the number of plies, I is the moment of inertia and A is the deformed cross sectional area. The element of the transformed reduced stiffness matrix $\bar{Q}_{ij}^{(k)}$, for a ply in its material coordinate system, is obtained as follows:

$$\begin{aligned} \bar{Q}_{11} &= C_{11}\cos^4\theta + 2\,(C_{12} + 2C_{66})\sin^2\theta\cos^2\theta + C_{22}\sin^4\theta \\ \bar{Q}_{66} &= (C_{11} + C_{22} - 2C_{12} - 2C_{66})\sin^2\theta\cos^2\theta + C_{66}\,(\sin^4\theta + \cos^4\theta) \end{aligned} \tag{9}$$

where θ is the angle between the global axis and the local axis of each layer, and the value of the angle θ is either 0^0 or 90^0 for cross-ply laminate. C_{ij} is the ply stiffness and it is given in terms of engineering constant of the k^{th} ply as:

$$C_{11} = \frac{E_1}{1-\upsilon_{12}\upsilon_{21}} \tag{10}$$

$$C_{12} = \frac{\upsilon_{12}E_2}{1-\upsilon_{12}\upsilon_{21}} = \frac{\upsilon_{21}E_1}{1-\upsilon_{12}\upsilon_{21}} \tag{11}$$

$$C_{22} = \frac{E_2}{1-\upsilon_{12}\upsilon_{21}} \tag{12}$$

$$C_{66} = G_{12} \tag{13}$$

where E_i is the Young's modulus in the i^{th} material direction, G_{ij} is the shear modulus of the i-j plane, and υ_{ij} is the Poisson ratio. Dynamic boundary conditions are given by Eq. (14), and geometric boundary conditions are given by Eq. (15):

$$\begin{aligned} -H &= -\hat{H} \\ -M &= -\hat{M} \end{aligned} \tag{14}$$

$$\begin{aligned} v &= \hat{v} \\ \Omega &= \hat{\Omega} \end{aligned} \tag{15}$$

In Eqs. (14) and (15), the quantities with hat are given at the boundary points, while quantities without hat are unknowns.

After the field equations are written in operator form as **Q** = **Lu** - **f**, where **L** represents the coefficient matrix, **u** represents unknown vectors (**u** = {v, M} for Euler-Bernoulli beams, and **u** = {v, Ω, T, M} for Timoshenko beams), and **f** represents the load vector. A necessary and sufficient condition for making the operator **Q** a potential is given by [26]. After the potentiality requirement is satisfied, the functional corresponds to the field equations and can be expressed as:

$$I(\mathbf{u}) = \int_0^1 \langle \mathbf{Q}\ (s\mathbf{u}), \mathbf{u} \rangle \, ds \tag{16}$$

Here, s is a scalar quantity and the parentheses <,> indicate the inner product. Considering the field equations of the laminated composite straight Euler-Bernoulli beam with boundary conditions, the matrix form of the Eq. (16) can be given by:

$$I(u) = \int_0^1 \left\{ \left[\begin{array}{cccc|cccc} -N\frac{d^2}{dz^2} & 0 & \frac{d^2}{dz^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{d^2}{dz^2} & 0 & \frac{1}{D_x} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right] \begin{bmatrix} sv \\ s\Omega \\ sM \\ sT \\ \hline sv \\ s\Omega \\ sM \\ sH \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \hline -\hat{H} \\ -\hat{M} \\ \hat{\Omega} \\ \hat{v} \end{bmatrix} \right\} \begin{bmatrix} v \\ 0 \\ M \\ 0 \\ \hline v \\ \Omega \\ M \\ H \end{bmatrix} ds$$

After Eq. (16) is implemented, the following expression is derived

$$I(u) = \int_0^1 \left(\begin{array}{l} -Nsv''v + sM''v + sv''M + \frac{1}{D_x} sMM - sHv + \hat{H}v - sM\Omega + \hat{M}\Omega \\ + s\Omega M - \hat{\Omega}M + svH - \hat{v}H \end{array} \right) ds$$

After integration and simplification, the explicit form of the functional corresponding to the field equations of the laminated composite straight Euler-Bernoulli beam is obtained as:

$$I(u) = \frac{1}{2}[Nv', v'] + \frac{1}{2D_x}[M, M] - [M', v'] + [\hat{H}, v]_\sigma + [v', M]_\sigma - [\hat{M}, v']_\sigma + [\hat{v}', M]_\varepsilon - [\hat{v}, H]_\varepsilon + [H, v]_\varepsilon \tag{17}$$

where [,] is the inner product defined as $[f,g] = \int fg dA$. The parentheses with subscripts ε and σ indicate the geometric and the dynamic boundary conditions, respectively. The prime symbol (') is used to represent the first derivative. The same mathematical procedure (i.e., applied for the functional of Euler-Bernoulli beam) is repeated for derivation of the functional corresponding to the laminated composite straight Timoshenko beams. For the sake of simplicity, the mathematical details for generating the functional of Timoshenko beams are not given here, but the interested reader is referred to [27]. The explicit expression of the functional of the laminated composite straight Timoshenko beams can be obtained in a similar manner as follows:

$$I(u) = \frac{1}{2}[Nv', v'] + \frac{1}{2D_x}[M, M] + \frac{1}{2C_y}[T, T] - [T, \Omega] - [M, \Omega'] - [v', T] + [\Omega, \hat{M}]_\sigma + [v, \hat{H}]_\sigma + [M, (\Omega - \hat{\Omega})]_\varepsilon + [H, (v - \hat{v})]_\varepsilon \tag{18}$$

3. Finite element formulation

If a one dimensional element with a parent shape function is used as follows:

$$\Psi_i = \frac{z_j - z}{L_e}, \qquad \Psi_j = \frac{z - z_i}{L_e} \tag{19}$$

where the adopted notation is illustrated in Figure 3, and L_e represents the length of an element ($L_e = z_j - z_i$), then all unknown variables of the functional given by Eq. (17) for Euler-Bernoulli beams are expressed in terms of interpolation functions as follows:

$$v = v_i\Psi_i + v_j\Psi_j \tag{20}$$

$$M = M_i\Psi_i + M_j\Psi_j \tag{21}$$

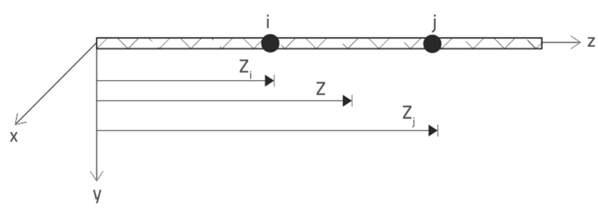

Figure 3. Two node one-dimensional element

After the functional is extremized with respect to the nodal variables, the following element matrix is derived explicitly as given by Eq. (22):

$$\left[\begin{array}{cc|cc} \frac{N}{L_e} & -\frac{1}{L_e} & -\frac{N}{L_e} & \frac{1}{L_e} \\ -\frac{1}{L_e} & \frac{L_e}{3D_x} & \frac{1}{L_e} & \frac{L_e}{6D_x} \\ \hline -\frac{N}{L_e} & \frac{1}{L_e} & \frac{N}{L_e} & -\frac{1}{L_e} \\ \frac{1}{L_e} & \frac{L_e}{6D_x} & -\frac{1}{L_e} & \frac{L_e}{3D_x} \end{array} \right] \begin{bmatrix} v_i \\ M_i \\ \hline v_j \\ M_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \hline 0 \\ 0 \end{bmatrix} \tag{22}$$

All unknown variables of the functional given by Eq. (18) for Timoshenko beams are expressed by shape functions as follows:

$$v = v_i\Psi_i + v_j\Psi_j \tag{23}$$

$$\Omega = \Omega_i\Psi_i + \Omega_j\Psi_j \tag{24}$$

$$M = M_i\Psi_i + M_j\Psi_j \tag{25}$$

$$T = T_i\Psi_i + T_j\Psi_j \tag{26}$$

When the functional is extremized with respect to nodal variables, the element matrix of Timoshenko beams is obtained as:

$$\left[\begin{array}{cccc|cccc} \frac{N}{L_e} & 0 & 0 & \frac{1}{2} & -\frac{N}{L_e} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{L_e}{3} & 0 & 0 & \frac{1}{2} & -\frac{L_e}{6} \\ 0 & \frac{1}{2} & \frac{L_e}{3D_x} & 0 & 0 & -\frac{1}{2} & \frac{L_e}{6D_x} & 0 \\ \frac{1}{2} & -\frac{L_e}{3} & 0 & \frac{L_e}{3C_y} & -\frac{1}{2} & -\frac{L_e}{6} & 0 & \frac{L_e}{6C_y} \\ \hline -\frac{N}{L_e} & 0 & 0 & -\frac{1}{2} & \frac{N}{L_e} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{L_e}{6} & 0 & 0 & -\frac{1}{2} & -\frac{L_e}{3} \\ 0 & \frac{1}{2} & \frac{L_e}{6D_x} & 0 & 0 & -\frac{1}{2} & \frac{L_e}{3D_x} & 0 \\ \frac{1}{2} & -\frac{L_e}{6} & 0 & \frac{L_e}{6C_y} & -\frac{1}{2} & -\frac{L_e}{3} & 0 & \frac{L_e}{3C_y} \end{array}\right] \left[\begin{array}{c} v_i \\ \Omega_i \\ M_i \\ T_i \\ \hline v_j \\ \Omega_j \\ M_j \\ T_j \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \tag{27}$$

4. Buckling analysis

An alternate form of buckling analysis problem can be given by:

$$\left(\begin{bmatrix} [K_{11}] & [K_{12}] \\ [K_{21}] & [K_{22}] \end{bmatrix} - P_{cr} \begin{bmatrix} [0] & [0] \\ [0] & [K_g] \end{bmatrix}\right) \begin{Bmatrix} \{F\} \\ \{v\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} \tag{28}$$

where {**F**} defines the stress resultant vector, {**v**} denotes the displacement vector, [**K**] is the system matrix, and [$\mathbf{K_g}$] is the geometric matrix of the system. Elimination of {**F**} from Eq. (28) yields the following equation:

$$\left([K^*] - P_{cr}[K_g]\right)\{v\} = \{0\} \tag{29}$$

where P_{cr} is defined as the critical buckling load and [**K***] is defined as the reduced system matrix of the problem.

$$[K^*] = [K_{22}] - [K_{21}][K_{11}]^{-1}[K_{12}] \tag{30}$$

The eigenvalues, P_{cr}, for which the determinant of coefficient matrix from Eq. (29) is zero, lead to the critical buckling loads.

5. Numerical examples

Example 1: Composite laminated Euler-Bernoulli beam
A symmetric cross-ply laminated composite beam with different number of layers and of equal thickness is considered.
The material properties can be given by:

E_1 = 155 GPa, E_2 = 12,1 GPa, υ_{12} = 0,248, G_{12} = 4,4 GPa

Data related to the beam:
Length: L = 0,25 m
Width: b = 0,01 m
Thickness of the beam: h = 0,001 m.
The beam is characterized by 20 equal length beam finite elements.

Different numerical examples are employed in order to test performance of the proposed method. First three modes of buckling loads (Newton "N"), P_{cr1}, P_{cr2} and P_{cr3} are compared with those available in the literature [9] as shown in Tables 1-3, using different boundary conditions. It can be seen from Tables 1-3 that the results of this study agree well with literature information, and so the methodology presented in this study is considered to be reliable. Comparing Tables 1-3, the beam with two ends hinged exhibits minimum

Table 1. Critical buckling loads (in [N]) of symmetric laminated composite beam with two ends hinged

Mode	(0°)		(90°)		(0°/90°/90°/0°)
	Present research	Research [9]	Present research	Research [9]	Present research
P_{cr1}	20.438	20.495	1.595	1.599	18.086
P_{cr2}	82.261	81.982	6.421	6.399	72.784
P_{cr3}	186.996	184.460	14.597	14.399	165.447
Mode	**(90°/0°/0°/90°)**	**(0°/90°/90°/90°/90°/0°)**		**(90°/90°/0°/0°/90°/90°)**	
	Present research	**Present research**	**Research [9]**	**Present research**	**Research [9]**
P_{cr1}	3.954	14.847	14.896	2.292	2.299
P_{cr2}	15.901	59.789	59.587	9.230	9.199
P_{cr3}	36.147	135.913	134.072	20.982	20.698

Table 2. Critical buckling loads (in [N]) of symmetric laminated composite beam with two ends clamped

Mode	(0°)		(90°)		(0°/90°/90°/0°)
	Present research	Research [9]	Present research	Research [9]	Present research
P_{cr1}	82.266	81.982	6.421	6.399	72.776
P_{cr2}	169.733	167.715	13.249	13.092	150.173
P_{cr3}	337.227	327.929	26.325	25.599	298.366
Mode	(90°/0°/0°/90°)	(0°/90°/90°/90°/90°/0°)		(90°/90°/0°/0°/90°/90°)	
	Present research	Present research	istraživanje [9]	Present research	Research [9]
P_{cr1}	15.903	59.777	59.587	9.228	9.199
P_{cr2}	32.81	123.364	121.901	19.045	18.819
P_{cr3}	65.187	245.105	238.350	37.840	36.797

Table 3. Critical buckling loads (in [N]) of symmetric laminated composite beam with one end clamped and another end hinged

Mode	(0°)		(90°)		(0°/90°/90°/0°)
	Present research	Research [9]	Present research	Research [9]	Present research
P_{cr1}	41.899	41.928	3.271	3.273	37.077
P_{cr2}	124.88	123.933	9.748	9.674	110.483
P_{cr3}	251.869	246.912	19.661	19.275	222.844
Mode	(90°/0°/0°/90°)	(0°/90°/90°/90°/90°/0°)		(90°/90°/0°/0°/90°/90°)	
	Present research	Present research	Research [9]	Present research	Research [9]
P_{cr1}	8.103	30.445	30.475	4.700	4.704
P_{cr2}	24.139	90.761	90.078	14.012	13.906
P_{cr3}	48.687	183.064	179.464	28.262	27.706

critical buckling loads, while the beam with two ends clamped exhibits maximum critical buckling load. The buckling loads of (0°), (0°/90°/90°/0°) and (0°/90°/90°/90°/90°/0°) beams are very high when compared to (90°), (90°/0°/0°/90°) and (90°/90°/0°/0°/90°/90°) beams. Here, a degree (°) refers to layer fibre orientation.

Example 2: Composite laminated Euler-Bernoulli beam

A symmetric laminated cross-ply composite beam with different number of layers of equal thickness is considered.

The material properties can be given by:

$E_1/E_2 = 25$, $E_3 = E_2$, $G_{12} = G_{13} = 0{,}5E_2$, $G_{23} = 0{,}2E_2$, $\upsilon_{12} = 0{,}25$.

Data related to the beam:

Length: L = 10 m

Width: b = 1 m

Thickness of the beam: h = 1 m.

The beam is divided into 20 finite elements of equal length. In this example, the dimensionless critical buckling load of

Table 4. Dimensionless critical buckling load ($\bar{P}_{cr}$) of symmetric laminated beam with H-H

Mode	(0°)		(90°)		(0°/90°/90°/0°)	
	Present research	Research [2]	Present research	Research [2]	Present research	Research [2]
$\bar{P}_{cr1}$	20.580	20.562	0.824	0.822	18.142	18.127
$\bar{P}_{cr2}$	82.923	-	3.317	-	72.974	-
$\bar{P}_{cr3}$	188.504	-	7.540	-	165.884	-
Mode	(90°/0°/0°/90°)		(0°/90°/90°/90°/90°/0°)		(90°/90°/0°/0°/90°/90°)	
	Present research	Research [2]	Present research		Present research	
$\bar{P}_{cr1}$	3.293	3.296	14.735		6.690	
$\bar{P}_{cr2}$	13.267	-	59.337		26.905	
$\bar{P}_{cr3}$	30.160	-	134.885		61.159	

Table 5. Dimensionless critical buckling load ($\bar{P}_{cr}$) of symmetric laminated beam with C-C

Mode	(0°)		(90°)		(0°/90°/90°/0°)	
	Present research	Research [2]	Present research	Research [2]	Present research	Research [2]
$\bar{P}_{cr1}$	82.904	82.247	3.317	3.290	72.989	72.507
$\bar{P}_{cr2}$	171.096	-	6.844	-	150.57	-
$\bar{P}_{cr3}$	339.941	-	13.597	-	299.153	-
Mode	**(90°/0°/0°/90°)**		**(0°/90°/90°/90°/90°/0°)**		**(90°/90°/0°/0°/90°/90°)**	
	Present research	**Research [2]**	**Present research**		**Present research**	
$\bar{P}_{cr1}$	13.265	13.183	59.335		26.905	
$\bar{P}_{cr2}$	27.376	-	122.434		55.513	
$\bar{P}_{cr3}$	54.391	-	243.251		110.293	

Table 6. Dimensionless critical buckling load ($\bar{P}_{cr}$) of symmetric laminated beam with C-F

Mode	(0°)		(90°)		(0°/90°/90°/0°)	
	Present research	Research [2]	Present research	Research [2]	Present research	Research [2]
$\bar{P}_{cr1}$	5.134	5.140	0.206	0.205	4.505	4.532
$\bar{P}_{cr2}$	46.45	-	1.859	-	40.913	-
$\bar{P}_{cr3}$	130.161	-	5.206	-	114.554	-
Mode	**(90°/0°/0°/90°)**		**(0°/90°/90°/90°/90°/0°)**		**(90°/90°/0°/0°/90°/90°)**	
	Present research	**Research [2]**	**Present research**		**Present research**	
$\bar{P}_{cr1}$	0.822	0.824	3.671		1.675	
$\bar{P}_{cr2}$	7.433	-	33.239		15.084	
$\bar{P}_{cr3}$	20.825	-	93.139		42.233	

the first three mode are compared with those available in the literature [2]. The dimensionless equation of critical buckling load is:

$$\bar{P}_{cr} = \frac{P_{cr} L^2}{b E_2 h^3} \quad (31)$$

Tables 4 to 6 present dimensionless critical buckling loads for hinged-hinged (H-H), clamped-clamped (C-C) and clamped-free (C-F) boundary conditions. Six lamination types are presented. As seen in the tables, the present model yielded results that in good agreement with the results given in [2].

Example 3- Composite laminated Euler-Bernoulli beam

In this example, three symmetric cross-ply laminated composite beams with four layers (0°/90°/90°/0°) and of equal thickness are considered. The cross sectional properties of the beams with two ends hinged are given as follows: the length of the beam is L = 10 m, the width of the beam is b = 1 m and the thickness of the beam is h = 1 m.

Different materials are considered in Table 7 to study the effect of material properties on critical buckling load of the first three mode of composite laminated beams.

Dimensionless critical buckling loads of composite laminated Euler-Bernoulli beams with (0°/90°/90°/0°) lamination are presented in Table 8.

Table 7. Material properties

	E_1 [GPa]	E_2 [GPa]	G_{12} [GPa]	G_{23} [GPa]	υ_{12}
Graphite epoxy (Grap)	181	10.3	7.17	3.433	0.28
Kevlar epoxy (Kev)	76	5.56	2.3	1.618	0.34

Table 8. Dimensionless critical buckling load ($\bar{P}_{cr}$) of symmetric cross-ply laminated Euler-Bernoulli beam (0°/90°/90°/0°) for different material properties

	Kev/Kev/Kev/Kev	Kev/Grap/Grap/Kev	Grap/Kev/Kev/Grap
$\bar{P}_{ü}$	159.547	161.049	377.015
$\bar{P}_{cr2}$	641.419	647.066	1518.304
$\bar{P}_{cr3}$	1458.035	1470.891	3451.526

It can be seen in Table 8 that the symmetric cross-ply laminated beam (0° Graphite/90° Kevlar/90° Kevlar/0° Graphite) exhibits the maximum critical buckling load whereas the beam (0° Kevlar/90° Kevlar/90° Kevlar/0° Kevlar) exhibits the minimum critical buckling load.

Example 4:Composite laminated Timoshenko beam

In this example, cross-ply laminated Timoshenko beams with (0°), (90°) and (90°/0°/0°/90°) lamination are considered. The material properties are as follows:

$$E_1/E_2 = 25,\ E_2 = E_3,\ G_{12} = G_{13} = 0{,}5E_2,\ G_{23} = 0{,}2E_2,\ \upsilon_{12} = 0{,}25.$$

Data related to the beam:
Length: L = 10 m
Width: b = 1 m
Thickness of the beam: h = 1 m.
The beam is characterized by 20 equal length beam finite elements.

This example is considered to test the method proposed in this research for Timoshenko beam. Dimensionless critical buckling loads of the symmetric cross-ply laminated beam with different boundary conditions are compared with those available in the literature. The comparison of results is given in Table 9. It can be seen from Table 9 that the results of this study agree well with literature information, and so the methodology presented in this study is considered reliable. The beam with one end clamped and another end free exhibits the minimum critical buckling load as compared to the boundary conditions C-C and H-H.

Example 5: Composite laminated Timoshenko beam

Dimensionless critical buckling load of a symmetric cross-ply laminated beam (0°/90°/0°) is considered in this example for different length-to-thickness ratios L/h = 5 and L/h = 10. The beam is divided into 20 finite elements of equal length. The corresponding material parameters are:

$$E_1/E_2 = 40,\ E_3 = E_2,\ G_{12} = G_{13} = 0{,}6E_2,\ G_{23} = 0{,}5E_2,\ \upsilon_{12} = 0{,}25.$$

Dimensionless critical buckling loads of the cross-ply laminated beam with different boundary conditions are compared to [11] as presented in Table 10. It can be seen in Table 10 that the results obtained in this study show good agreement with literature. In addition, the critical buckling load of the laminated beam increases with an increase in the length-to-thickness ratio as expected. Furthermore, the beam with C-F exhibits the minimum critical buckling load whereas the beam with C-C exhibits the maximum critical buckling load.

Table 10. Dimensionless critical buckling load ($\bar{P}_{cr}$) of symmetric cross-ply laminated beam (0°/90°/0°) for different length-to-thickness ratios

L/h	Boundary condition	(0°/90°/0°)	
		Present research	Research [11]
5	H-H	8.588	8.606
	C-H	9.386	9.412
	C-C	10.752	10.802
	C-F	4.821	4.747
10	H-H	18.932	18.989
	C-H	25.840	25.940
	C-C	34.845	34.426
	C-F	6.793	6.797

6. Conclusion

The buckling analysis of symmetric cross-ply laminated composite beams is carried out using the MFE method. Two different beam theories, i.e. the composite laminated Euler-Bernoulli beam theory and the composite laminated Timoshenko beam theory, are considered. Two new functionals for the buckling analysis of symmetric cross-ply laminated composite Euler-Bernoulli and Timoshenko beams are established based on the Gâteaux differential approach. Employing the developed finite element formulation, symmetric cross-ply laminated composite beams

Table 9. Dimensionless critical buckling load ($\bar{P}_{cr}$) of symmetric cross-ply laminated beam L/h = 10 with different boundary conditions

L/h	Boundary condition	(0°)		(90°)		(90°/0°/0°/90°)	
		Present research	Research [2]	Present research	Research [2]	Present research	Research [2]
10	H-H	13,739	13,768	0,782	0,784	11,889	11,179
	C-C	27,359	27,656	2,727	2,747	20,940	20,800
	C-F	4,666	4,576	0,267	0,203	4,770	3,922

including different number of layers are considered as examples for numerical evaluation of the effects of the beam theories, variation of geometrical and material parameters, and boundary conditions, on the critical buckling load. Comparison between results obtained in this study and literature results reveals that they are in good agreement. Results given in numerical examples point to the validity and efficiency of the presented formulation. The following main conclusions may be drawn:

- Overestimation of beam stiffness with the Euler-Bernoulli beam theory leads to the critical buckling load being larger when compared to the Timoshenko beam theory.
- Considering the same types of lamination (0^0), (90^0) and ($90^0/0^0/0^0/90^0$), it has been demonstrated that the laminated Euler-Bernoulli and Timoshenko beams with two ends clamped have the maximum critical buckling load.
- The difference between the critical buckling loads of the Euler-Bernoulli and Timoshenko beam theories decreases with an increase in the length to thickness ratio.
- The critical buckling loads of the symmetric laminated Euler-Bernoulli beam and Timoshenko beam differ from one another, and the difference between the values predicted by the two theories becomes more significant for the beam with ($90^0/0^0/0^0/90^0$) lamination.
- Considering four lamination types, i.e. ($0^0/90^0/90^0/0^0$), ($0^0/90^0/90^0/90^0/90^0/0^0$), ($90^0/0^0/0^0/90^0$), and ($90^0/90^0/0^0/0^0/90^0/90^0$), it has been proven that the stiffness of the symmetric laminated Euler-Bernoulli beam with ($0^0/90^0/90^0/0^0$) or ($0^0/90^0/90^0/90^0/90^0/0^0$) lamination is higher than ($90^0/0^0/0^0/90^0$) or ($90^0/90^0/0^0/0^0/90^0/90^0$) lamination, respectively.
- The MFE formulation is developed in this study for analysing buckling of symmetric cross-ply laminated composite straight Euler-Bernoulli and Timoshenko beams, utilizing new functionals through a systematic procedure based on the Gâteaux differential.
- The Gâteaux differential approach is reliable and very simple to implement, and its results are reasonably accurate for engineering purposes (i.e. this approach is capable of predicting displacements and internal forces directly without requiring mathematical operations). The same approach can be applied for the buckling and vibration analysis of cross-ply and/or angle-ply laminated composite beams by considering higher-order shear deformation theories. Some of these problems are currently under study in accordance with the methodology presented in the paper.

REFERENCES

[1] Jones, M.R.: Mechanics of Composite Materials, Taylor & Francis, 1999.

[2] Reddy, J.N.: Mechanics of Laminated Composite Plates and Shells: Theory and Analysis, Second edition, CRC Press, 2004.

[3] Kaw, A.K.: Mechanics of Composite Materials, Taylor & Francis, 2006.

[4] Aydoğdu, M.: Buckling analysis of cross-ply laminated beams with general boundary conditions by Ritz method, Composites Science and Technology, 66 (2006), pp. 248–1255, https://doi.org/10.1016/j.compscitech.2005.10.029

[5] Jun, L., Hongxing, H., Rongying, S.: Dynamic stiffness analysis for free vibrations of axially loaded laminated composite beams, Composite Structures, 84 (2008), pp. 87-98, https://doi.org/10.1016/j.compstruct.2007.07.007

[6] Jun, L., Xiaobin, L., Hongxing, H.: Free vibration analysis of third-order shear deformable composite beams using dynamic stiffness method, Archive of Applied Mechanics, 79 (2009), pp. 1083-1098, https://doi.org/10.1007/s00419-008-0276-8

[7] Jun, L., Hongxing, H.: Free vibration analyses of axially loaded laminated composite beams based on higher-order shear deformation theory, Meccanica, 46 (2011), pp. 1299-1317, https://doi.org/10.1007/s11012-010-9388-7

[8] Xu, R., Wu, Y.: Free vibration and buckling of composite beams with interlayer slip by two-dimensional theory, Journal of Sound and Vibration, 313 (2008), pp. 875–890, https://doi.org/10.1016/j.jsv.2007.12.029

[9] Emam, S.A., Nayfeh, A.H.: Postbuckling and free vibrations of composite beams, Composite Structure, 88 (2009), pp. 636-642, https://doi.org/10.1016/j.compstruct.2008.06.006

[10] Khdeir, A.A., Reddy, J.N.: Free vibration of cross-ply laminated beams with arbitrary boundary conditions, International Journal of Engineering Science, 32 (1994) 12, pp. 1971–1980, https://doi.org/10.1016/0020-7225(94)90093-0

[11] Khdeir, A.A., Reddy, J.N.: Buckling of cross-ply laminated beams with arbitrary boundary conditions, Composite Structures, 37 (1997) 1, pp. 1-3, https://doi.org/10.1016/S0263-8223(97)00048-2

[12] Loja, M.A.R, Barbosa, J.I., Soares, C.M.M.: Buckling behavior of laminated beam structures using a higher-order discrete model, Composite Structures, 38 (1997) 1-4, pp. 119-131, https://doi.org/10.1016/S0263-8223(98)80011-1

[13] Xu, R., Wu, Y.: Static, dynamic, and buckling analysis of partial interaction composite members using Timoshenko's beam theory, International Journal of Mechanical Sciences, 49 (2007) 10, pp. 1139–1155, https://doi.org/10.1016/j.ijmecsci.2007.02.006

[14] Abadi, M.M., Daneshmehr, A.R.: An investigation of modified couple stress theory in buckling analysis of micro composite laminated Euler–Bernoulli and Timoshenko beams, International Journal of Engineering Science, 75 (2014), pp. 40–53, https://doi.org/10.1016/j.ijengsci.2013.11.009

[15] Matsunaga, H.: Vibration and buckling of multilayered composite beams according to higher order deformation theories, Journal of Sound and Vibration, 246 (2001), pp. 47-62, https://doi.org/10.1006/jsvi.2000.3627

[16] Zhen, W., Wanji, C.: An assessment of several displacement-based theories for the vibration and stability analysis of laminated composite and sandwich beams, Composite Structures, 84 (2008) 4, pp. 337-349, https://doi.org/10.1016/j.compstruct.2007.10.005

[17] Vo, T., Inam, F.: Vibration and buckling of cross-ply composite beams using refined shear deformation theory, 2nd International Conference on Advanced Composite Materials and Technologies for Aerospace Applications, UK, 2012.

[18] Ibrahim, S.M., Carrera, E., Petrolo, M., Zappino, E.: Buckling of composite thin walled beams by refined theory, Composite Structures, 94 (2012) 2, pp. 563-570, https://doi.org/10.1016/j.compstruct.2011.08.020

[19] Vo, T.P., Thai, H-T.: Vibration and buckling of composite beams using refined shear deformation theory, International Journal of Mechanical Sciences, 62 (2012) 1, pp. 67-76, https://doi.org/10.1016/j.ijmecsci.2012.06.001

[20] He, G., Yang, X.: Finite element analysis for buckling of two-layer composite beams using Reddy's higher order beam theory, Finite Elements in Analysis and Design, 83 (2014), pp. 49-57, https://doi.org/10.1016/j.finel.2014.01.004

[21] Aköz, A.Y.: A new functional for bars and its applications, 4th National Applied Mechanics Meeting, Turkey, 1985.

[22] Kadıoğlu, F., Iyidogan, C.: Free vibration of laminated composite curved beams using mixed finite element formulation, Science and Engineering of Composite Materials,16 (2009) 4, pp. 247-257, https://doi.org/10.1515/SECM.2009.16.4.247

[232] Özütok, A., Madenci, E.: Free vibration analysis of cross-ply laminated composite beams by using mixed finite element formulation, International Journal of Structural Stability and Dynamics, 13 (2013) 2, 1250056-17, https://doi.org/10.1142/S0219455412500563

[24] Özütok, A., Madenci, E., Kadıoğlu, F.: Free vibration analysis of angle-ply laminate composite beams by mixed finite element formulation using the Gâteaux differential, Science and Engineering of Composite Materials, 21 (2014) 2, pp. 257-266, https://doi.org/10.1515/secm-2013-0043

[25] Ateș, N., Tekin, G., Kadıoğlu, F.: Alternative Solution for Cross-ply Laminated Composite Thick Plates, Građevinar, Accepted, 2016.

[26] Oden, J.T. Reddy, J.N.: Variational methods in theoretical mechanics, Springer, Berlin, 1976, https://doi.org/10.1007/978-3-642-96312-4

[27] Özkan, G.: Buckling analysis of cross-ply laminated composite straight beams, M.Sc. Dissertation, Department of Civil Engineering. Istanbul Technical University, Istanbul, 2011.